

# Two and three-dimensional oscillons in nonlinear Faraday resonance

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We study 2D and 3D localised oscillating patterns in a simple model system exhibiting nonlinear Faraday resonance. The corresponding amplitude equation is shown to have exact soliton solutions which are found to be always unstable in 3D. On the contrary, the 2D solitons are shown to be stable in a certain parameter range; hence the damping and parametric driving are capable of suppressing the nonlinear blowup and dispersive decay of solitons in two dimensions. The negative feedback loop occurs via the enslaving of the soliton's phase, coupled to the driver, to its amplitude and width.

Oscillons are localised two-dimensional oscillating structures which have recently been detected in experiments on vertically vibrated layers of granular material [1], Newtonian fluids and suspensions [2,3]. Numerical simulations established the existence of stable oscillons in a variety of pattern-forming systems, including the Swift-Hohenberg and Ginzburg-Landau equations, period-doubling maps with continuous spatial coupling, semicontinuum theories and hydrodynamic models [4,3]. Although these simulations provided a great deal of insight into the phenomenology of the oscillons (in particular, demarcated their existence area on the corresponding phase diagrams), little is known about the mechanism by which they acquire or loose their stability.

In this Letter, we consider a model equation which has *exact* oscillon solutions and allows an accurate characterisation of their existence and stability domains. The main purpose of this work is to understand how the oscillons manage to resist the general tendencies toward nonlinearity-induced blow-up or dispersive decay which are characteristic for localised excitations in two-dimensional media. Our model admits a straightforward generalisation to three dimensions and we use this opportunity to explore the existence of stable oscillons in 3D as well.

The model consists of a  $D$ -dimensional lattice of parametrically driven nonlinear oscillators (e.g. pendula) [5] with the nearest-neighbour coupling:

$$\frac{d^2}{d\tau^2}\phi_{\mathbf{k}} + \alpha \frac{d}{d\tau}\phi_{\mathbf{k}} + 2\kappa D \phi_{\mathbf{k}} - \kappa \sum_{|\mathbf{m}-\mathbf{k}|=1} \phi_{\mathbf{m}} + (1 + \rho \cos 2\omega\tau) \sin \phi_{\mathbf{k}} = 0; \quad \mathbf{k} = (k_1, \dots, k_D). \quad (1)$$

Assuming that the coupling is strong:  $\kappa \gg 1$ ; that the damping and driving are weak:  $\alpha = \gamma\varepsilon^2$ ,  $\rho = 2h\varepsilon^2$  where  $\varepsilon \ll 1$ ; and that the driving half-frequency is just below the edge of the linear spectrum gap:  $\omega^2 = 1 - \varepsilon^2$ , the oscillators execute small-amplitude librations of the form  $\phi_{\mathbf{k}} = 2\varepsilon\psi(t, \mathbf{x}_{\mathbf{k}})e^{-i\omega\tau} + c.c. + O(\varepsilon^3)$ , where  $t = \varepsilon^2\tau/2$ ,  $\mathbf{x}_{\mathbf{k}} = \frac{\varepsilon}{\sqrt{\kappa}}\mathbf{k}$  and the slowly varying amplitude satisfies

$$i\psi_t + \nabla^2\psi + 2|\psi|^2\psi - \psi = h\psi^* - i\gamma\psi, \quad (2)$$

the parametrically driven damped nonlinear Schrödinger (NLS) equation. In 2D, this equation was invoked as a phenomenological model of nonlinear Faraday resonance in water [3]. It also describes an optical resonator with different losses for the two polarisation components of the field [6]. In the absence of the damping and driving, all localised initial conditions in the 2D and 3D NLS equation are known to either disperse or blow-up in finite time [7–9]. Surprisingly, numerical simulations of (2) with sufficiently large  $h$  and  $\gamma$  revealed the occurrence of stable (or possibly long-lived) stationary localised excitations [3]. However no analytic solutions were found, and a possible stabilisation mechanism remained unclear.

In fact there are two exact (though not explicit) stationary radially-symmetric solutions given by

$$\psi^{\pm} = \mathcal{A}_{\pm} e^{-i\theta_{\pm}} \mathcal{R}_0(\mathcal{A}_{\pm} r); \quad (r^2 = x_1^2 + \dots + x_D^2), \quad (3)$$

where  $\mathcal{A}_{\pm}^2 = 1 \pm \sqrt{h^2 - \gamma^2}$ ,  $\theta_+ = \frac{1}{2} \arcsin(\gamma/h)$ ,  $\theta_- = \frac{\pi}{2} - \theta_+$ , and  $\mathcal{R}_0(r)$  is the bell-shaped nodeless solution of

$$\nabla_r^2 \mathcal{R} - \mathcal{R} + 2\mathcal{R}^3 = 0; \quad \mathcal{R}_r(0) = \mathcal{R}(\infty) = 0. \quad (4)$$

(Below we simply write  $\mathcal{R}$  for  $\mathcal{R}_0$ .) In (4),  $\nabla_r^2 = \partial_r^2 + (D-1)r^{-1}\partial_r$ . Solutions of Eq.(4) in  $D = 2$  and  $3$  are well documented in literature. (See e.g. [7] and refs therein.) One advantage of having an explicit dependence on  $h$  and  $\gamma$ , is that the existence domain is characterised by an explicit formula. The soliton  $\psi^+$  exists for all  $h > \gamma$ ; the  $\psi^-$  exists for  $\gamma < h < \sqrt{1 + \gamma^2}$ . It is pertinent to add here that for  $h < \gamma$ , *all* initial conditions decay to zero. This follows from the rate equation

$$\partial_t |\psi|^2 = \frac{2}{r} [r(\chi_r |\psi|^2)_r]_r + 2|\psi|^2 (h \sin 2\chi - \gamma), \quad (5)$$

where  $\psi = |\psi|e^{-i\chi}$ . Defining  $N = \int |\psi|^2 d\mathbf{x}$ , Eq.(5) implies  $N_t \leq 2(h - \gamma)N$  whence  $N(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

We now examine the stability of the two solitons. Linearising Eq.(2) in the small perturbation

$$\delta\psi(\mathbf{x}, t) = e^{(\mu - \Gamma)\tilde{t} - i\theta_{\pm}} [u(\tilde{\mathbf{x}}) + iv(\tilde{\mathbf{x}})], \quad (6)$$

where  $\tilde{\mathbf{x}} = \mathcal{A}_{\pm}\mathbf{x}$ ,  $\tilde{t} = \mathcal{A}_{\pm}^2 t$ , we get an eigenvalue problem

$$L_1 u = -(\mu + \Gamma)v, \quad (L_0 - \epsilon)v = (\mu - \Gamma)u, \quad (7)$$

where  $\Gamma = \gamma/\mathcal{A}_\pm^2$  and the operators

$$L_0 \equiv -\tilde{\nabla}^2 + 1 - 2\mathcal{R}^2(\tilde{r}), \quad L_1 \equiv L_0 - 4\mathcal{R}^2(\tilde{r}), \quad (8)$$

with  $\tilde{\nabla}^2 = \sum_{i=1}^D \partial^2/\partial \tilde{x}_i^2$ . (We are dropping tildas below.) The quantity  $\epsilon$ ,  $\epsilon = \pm 2\sqrt{h^2 - \gamma^2}/\mathcal{A}_\pm^2$ , is positive for the  $\psi^+$  soliton and negative for  $\psi^-$ . Each  $\epsilon$  defines a “parabola” on the  $(h, \gamma)$ -plane:

$$h = \sqrt{\epsilon^2/(2 - \epsilon)^2 + \gamma^2}. \quad (9)$$

Introducing  $\lambda^2 = \mu^2 - \Gamma^2$  and changing  $v(\mathbf{x}) \rightarrow (\mu + \Gamma)\lambda^{-1}v(\mathbf{x})$  [10], Eq.(7) is reduced to a *one*-parameter eigenvalue problem:

$$(L_0 - \epsilon)v = \lambda u, \quad L_1 u = -\lambda v. \quad (10)$$

Since  $\mathcal{R}_0(r)$  is nodeless in  $0 \leq r < \infty$ , and  $L_0 \mathcal{R}_0 = 0$ , the operator  $L_0 - \epsilon$  is positive definite for  $\epsilon < 0$ . In this case the eigenvalue can be found as a minimum of the Rayleigh quotient:

$$-\lambda^2 = \min_w \frac{\langle w | L_1 | w \rangle}{\langle w | (L_0 - \epsilon)^{-1} | w \rangle}. \quad (11)$$

The operator  $L_1$  has  $D$  zero eigenvalues associated with the translation eigenfunctions  $\partial_i \mathcal{R}(r)$ ,  $i = 1, 2, \dots, D$ ; hence it also has a negative eigenvalue with a radially-symmetric eigenfunction  $w_0(r)$ . Substituting  $w_0$  into the quotient in (11), we get  $-\lambda^2 < 0$  whence  $\mu > \Gamma$ . Thus the soliton  $\psi^-$  is unstable (against a nonoscillatory mode) for all  $D$ ,  $h$  and  $\gamma$ , and may be safely disregarded.

Before proceeding to the stability of  $\psi^+$  (for which we have  $\epsilon > 0$ ), we make a remark on the undamped, un-driven case ( $\epsilon = 0$ .) In 3D, the eigenvalue problem (10) has a zero eigenvalue associated with the phase invariance of the unperturbed NLS equation (2) and another one, associated with the scaling symmetry:

$$\begin{pmatrix} L_0 & 0 \\ 0 & L_1 \end{pmatrix} \begin{pmatrix} \mathcal{R} \\ -\frac{1}{2}(r\mathcal{R})_r \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{R} \end{pmatrix}. \quad (12)$$

Both the eigenvector  $(\mathcal{R}, 0)^T$  and the rank-2 generalised eigenvector  $(0, -\frac{1}{2}(r\mathcal{R})_r)^T$  are radially-symmetric. In 2D the number of repeated zero eigenvalues associated with radially-symmetric invariances is four; in addition to those in (12) we have a two-parameter group of the lens transformations [7,8] giving rise to

$$\begin{pmatrix} L_0 & 0 \\ 0 & L_1 \end{pmatrix} \begin{pmatrix} \frac{1}{8}r^2\mathcal{R} \\ g \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(r\mathcal{R})_r \\ \frac{1}{8}r^2\mathcal{R} \end{pmatrix}, \quad (13)$$

with some  $g(r)$ . When  $h^2 - \gamma^2$  (or, equivalently,  $\epsilon$ ) deviates from zero, all the above invariances break down and the two (respectively, four) eigenvalues move away from the origin on the plane of complex  $\lambda$ . The directions of their motion are crucial for the stability properties.

We can calculate  $\lambda(\epsilon)$  perturbatively, assuming

$$\lambda = \lambda_1 \epsilon^{\frac{1}{4}} + \lambda_2 \epsilon^{\frac{3}{4}} + \lambda_3 \epsilon^{\frac{5}{4}} + \dots, \quad (14)$$

$$u = u_1 \epsilon^{\frac{1}{4}} + u_2 \epsilon^{\frac{3}{4}} + \dots, \quad v = \mathcal{R} + v_1 \epsilon^{\frac{1}{4}} + v_2 \epsilon^{\frac{3}{4}} + \dots,$$

where  $v_i = v_i(r)$ ,  $u_i = u_i(r)$ . Substituting into (10), the order  $\epsilon^{1/4}$  gives  $u_1 = -\lambda_1 L_1^{-1} \mathcal{R}$ . Using (12),  $u_1$  is found explicitly:  $u_1 = (\lambda_1/2)(r\mathcal{R})_r$ . At the order  $\epsilon^{3/4}$  we get  $u_2 = -\lambda_2 L_1^{-1} \mathcal{R}$  and equation  $L_0 v_2 = \lambda_1 u_1$ . Since  $L_0$  has a null eigenvector,  $\mathcal{R}(r)$ , this equation is only solvable if

$$\lambda_1 \int \mathcal{R}(r) u_1(r) d\mathbf{x} = -\lambda_1^2 \frac{D-2}{4} \int \mathcal{R}^2(r) d\mathbf{x} = 0. \quad (15)$$

In the two-dimensional case the condition (15) is satisfied for any  $\lambda_1$  whereas in  $D = 3$  we have to set  $\lambda_1 = 0$ . Next, at the orders  $\epsilon^{3/4}$  and  $\epsilon^{5/4}$  we obtain, respectively,

$$L_0 v_3 = \lambda_2 u_1 + \lambda_1 u_2 = \lambda_1 \lambda_2 (r\mathcal{R})_r, \quad (16)$$

$$L_0 v_4 = R + \lambda_1 u_3 + \lambda_2 u_2 + \lambda_3 u_1. \quad (17)$$

Eq.(16) is solvable both in 2D and 3D. The solvability condition for (17) reduces to

$$\lambda_1^4 = -\frac{\langle \mathcal{R} | \mathcal{R} \rangle}{\langle \mathcal{R} | L_1^{-1} L_0^{-1} L_1^{-1} | \mathcal{R} \rangle} = -16 \frac{\int \mathcal{R}^2 d\mathbf{x}}{\int \mathcal{R}^2 r^2 d\mathbf{x}}, \quad (18)$$

$$\lambda_2^2 = \frac{\langle \mathcal{R} | \mathcal{R} \rangle}{\langle \mathcal{R} | L_1^{-1} | \mathcal{R} \rangle} = 4, \quad (19)$$

in two and three dimensions, respectively.

Thus we arrive at two different bifurcation scenarios. In 3D, where  $\lambda_1 = 0$  and  $\lambda_2$  is real, two imaginary eigenvalues  $\pm |\lambda_2| \epsilon^{1/2}$  converge at the origin as  $\epsilon \rightarrow 0$  from the left. (This does not mean that the  $\psi^-$  soliton is stable as there still is a pair of finite real eigenvalues for  $\epsilon < 0$ .) As  $\epsilon$  grows to positive values, the imaginary pair  $\pm |\lambda_2| \epsilon^{1/2}$  moves onto the real axis. A numerical study [11] of the eigenvalue problem (10) shows that when  $\epsilon$  is further increased, the four real eigenvalues collide, pairwise, and acquire imaginary parts. Importantly, for all  $0 < \epsilon < 1$  the imaginary parts remain smaller in magnitude than the real parts. This means that  $\text{Re} \mu$  remains greater than  $\Gamma$  all the time, implying that the three-dimensional  $\psi^+$  soliton is unstable for all  $h$  and  $\gamma$ .

The bifurcation occurring in 2D is more unusual. As  $\epsilon$  approaches zero from the left, *four* eigenvalues converge at the origin, two along the real and two along imaginary axis:  $\lambda \approx \pm |\lambda_1| (-\epsilon)^{1/4}$ ,  $\pm i |\lambda_1| (-\epsilon)^{1/4}$ . As  $\epsilon$  moves to positive, the four eigenvalues start diverging at  $45^\circ$  to the real and imaginary axes. Hence to the leading order,  $\text{Im} \lambda \approx \text{Re} \lambda$ , and in order to make a conclusion about the stability, we need to calculate the higher-order corrections. The order  $\epsilon^{5/4}$  produces a solvability condition

$$\lambda_1^3 \lambda_2 \langle \mathcal{R} | L_1^{-1} L_0^{-1} L_1^{-1} | \mathcal{R} \rangle = \frac{\lambda_1^3 \lambda_2}{16} \int \mathcal{R}^2 r^2 d\mathbf{x} = 0,$$

yielding  $\lambda_2 = 0$ . (Here we made use of (13).) Finally, the order  $\epsilon^{6/4}$  defines  $\lambda_3$  (where  $g(r)$  is as in (13)):

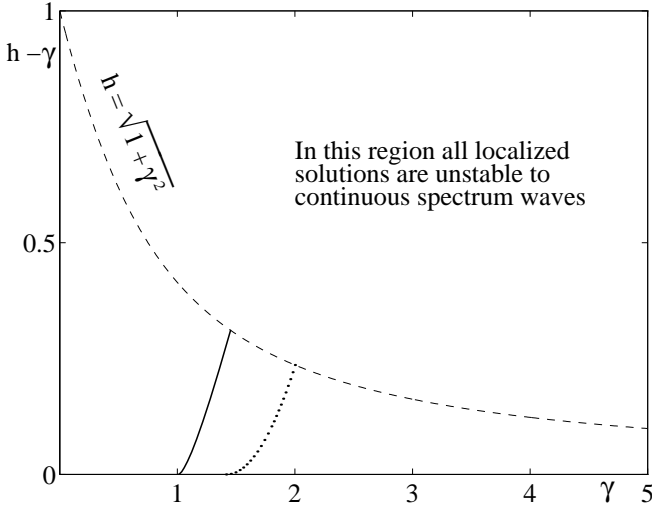


FIG. 1. Stability diagram for two-dimensional solitons. The  $(\gamma, h - \gamma)$ -plane is used for visual clarity. No localised or periodic attractors exist for  $h < \gamma$  (below the horizontal axis). The region of stability of the soliton  $\psi^+$  lies to the right of the solid curve. The dotted curve gives the variational approximation to the stability boundary of the  $\psi^+$  soliton:  $h = (1 + \gamma^4)^{1/2}$ ,  $\gamma \geq \sqrt{2}$ .

$$\lambda_3 = \frac{1}{\lambda_1} + \frac{\lambda_1^3}{2} \frac{\int g(r) \mathcal{R}(r) r^2 d\mathbf{x}}{\int \mathcal{R}^2(r) r^2 d\mathbf{x}}. \quad (20)$$

Taking  $\lambda_1$  in the first quadrant,  $\lambda_1 = e^{i\pi/4} |\lambda_1|$ , and doing the integrals in (18), (20) numerically, we conclude that  $\lambda_3$  is in the second quadrant,  $\lambda_3 = e^{3i\pi/4} |\lambda_3|$ , which implies that  $|\text{Im}\lambda| > |\text{Re}\lambda|$ . In terms of  $\lambda$ , the stability criterion  $\text{Re}\mu \leq \Gamma$  is written as  $\gamma \geq \gamma_c$ , where

$$\gamma_c(\epsilon) \equiv \frac{2}{2 - \epsilon} \cdot \frac{\text{Re}\lambda(\epsilon) \text{Im}\lambda(\epsilon)}{\sqrt{(\text{Im}\lambda)^2 - (\text{Re}\lambda)^2}}. \quad (21)$$

The smallest  $\gamma$  for which the soliton can be stable, is given by

$$\lim_{\epsilon \rightarrow 0} \gamma_c(\epsilon) = \frac{1}{2\sqrt{2}} |\lambda_1|^{3/2} |\lambda_3|^{-1/2}. \quad (22)$$

Substituting for  $\lambda_1, \lambda_3$  their numerical values, (22) gives  $\gamma_c(0) = 1.00647$ . For  $\epsilon \neq 0$  we obtained  $\lambda(\epsilon)$  by solving the eigenvalue problem (10) directly [11]. Here we have restricted ourselves to radially-symmetric  $u(r)$  and  $v(r)$ . Expressing  $\epsilon$  via  $\gamma_c$  from (21) and feeding into (9), we get the stability boundary on the  $(h, \gamma)$ -plane (Fig.1).

Asymmetric perturbations do not lead to any instabilities in 2D. To show this, we factorise, in (10),  $u(\mathbf{x}) = \tilde{u}(r)e^{im\varphi}$  and  $v(\mathbf{x}) = \tilde{v}(r)e^{im\varphi}$ , where  $\tan \varphi = y/x$  and  $m$  is an integer. The eigenproblem (10) remains the same, with only the operators  $L_0$  and  $L_1$  being replaced by

$$L_0^{(m)} \equiv L_0 + m^2/r^2, \quad L_1^{(m)} \equiv L_1 + m^2/r^2. \quad (23)$$

The crucial observation now is that  $L_0^{(m)}$  with  $m^2 \geq 1$  does not have *any* (not even positive) discrete eigenvalues. We verified this numerically for  $m^2 = 1$ ; this rules

out their appearance for all other  $m$ . Therefore the operator  $L_0^{(m)} - \epsilon$  with  $\epsilon < 1$  is positive definite, and the eigenvalues of the problem (10) can be found from the variational principle (11). The operator  $L_1^{(1)}$  has a zero eigenvalue with the eigenfunction  $w^{(1)}(r) = \mathcal{R}_r(r)$  which has no nodes for  $0 < r < \infty$ ; hence its all other eigenvalues (if exist) are positive. This also implies that  $L_1^{(m)}$  with  $m^2 > 1$  are positive definite. Thus the minimum of the Rayleigh quotient (11) is zero for  $m^2 = 1$  and positive for  $m^2 > 1$ .

Besides the nodeless solution  $\mathcal{R}_0(r)$ , the “master” equation (4) has solutions  $\mathcal{R}_n(r)$  with  $n$  nodes,  $n = 1, 2, \dots$ . These give rise to a sequence of nodal solutions of the damped-driven NLS (2), defined by Eq.(3) with  $\mathcal{R}_0 \rightarrow \mathcal{R}_n$ . It is easy to realise that the solitons  $\psi_n^-$  are unstable against radially-symmetric nonoscillatory modes for all  $h, \gamma, n$  and  $D$ . (The proof is a simple generalisation of the one for  $\psi_0^-$ .) To examine the stability of the  $\psi_n^+$  soliton, we solved the eigenvalue problem (10) numerically, with the operators  $L_{0,1}^{(m)}$  as in (23). In 3D, positive real eigenvalues (with radially-symmetric eigenfunctions) are present in the spectrum for all  $\epsilon$ ; thus the three-dimensional nodal solitons are always prone to a symmetric collapse or dispersive spreading. In 2D, the  $\psi_n^+$  solitons are stable against radially-symmetric perturbations for sufficiently large  $\gamma$ . However, these solutions turn out to be always unstable against azimuthal perturbations. In particular, the  $\psi_1^+$  soliton has instabilities associated with  $1 \leq m \leq 5$ , and the  $m = 4$  mode has the largest growth rate for all  $\epsilon$ . The corresponding eigenvalue  $\lambda$  is real and the eigenfunctions  $u(r)$  and  $v(r)$  have a single maximum near the position of the lateral minimum of the function  $R_1(r)$ . Following Ref. [12] where a similar scenario was described for nodal waveguides in a saturable self-focusing medium, the above observation suggests that the  $\psi_1^+$  soliton will decay into 5 solitons  $\psi_0^+$ : one at the origin and four others placed symmetrically around it. Next, the  $\psi_2^+$  solution has azimuthal instabilities with  $1 \leq m \leq 10$ . The analysis of the corresponding eigenfunctions suggests that, depending on  $h$  and  $\gamma$ , the decay products will comprise 11 to 13  $\psi_0^+$ -solitons: 1 at the origin; 3 or 4 placed symmetrically around it; and 7 or 8 forming an outer ring. We verified these predictions via direct numerical simulations of the time-dependent array (1); the simulations corroborated the above scenario. Thus the nodal solutions can be interpreted as degenerate coaxial complexes of the nodeless solitons and serve as nuclei of symmetric multisoliton patterns.

Lastly, we need to understand the stabilisation mechanism in qualitative terms. To this end, we use the variational approach. The equation (2) is derivable from the stationary action principle with the Lagrangian

$$\mathcal{L} = e^{2\gamma t} \text{Re} \int (i\psi_t \psi^* - |\nabla \psi|^2 - |\psi|^2 + |\psi|^4 - h\psi^2) d\mathbf{x}.$$

Choosing the ansatz  $\psi = \sqrt{A}e^{-i\theta-(B+i\sigma)r^2}$  [13,14] with  $A, B, \theta, \sigma$  functions of  $t$ , this reduces, in 2D, to

$$\mathcal{L} = e^{2\gamma t} \frac{A}{B} \left[ \dot{\theta} - 1 + \frac{\dot{\sigma}}{2B} - \frac{2B}{\cos^2 \phi} + \frac{A}{2} - h \cos(\phi + 2\theta) \cos \phi \right]; \quad \tan \phi = \sigma/B. \quad (24)$$

The 4-dimensional dynamical system defined by (24), has two stationary points representing the  $\psi^\pm$  solitons. In agreement with the stability properties of the solitons in the full PDE, the  $\psi^+$  stationary point is unstable for small  $\gamma$  but stabilises for larger dampings (Fig.1). When  $\gamma$  is large we can expand  $A = A_0 + \frac{1}{\gamma}A_1 + \dots$ ,  $B = B_0 + \frac{1}{\gamma}B_1 + \dots$ ,  $\theta = \frac{\pi}{4} + \frac{1}{\gamma}\theta_1 + \dots$ ,  $\sigma = \frac{1}{\gamma}\sigma_1 + \dots$ . Letting  $h = \gamma + \frac{c}{2\gamma}$  where  $0 \leq c \leq 1$ , defining  $T = \frac{t}{\gamma}$  and matching coefficients of like powers of  $\frac{1}{\gamma}$ , yields a 2-dimensional system

$$dA_0/dT = A_0[c + 8\sigma_1 - 4\theta_1^2 + 2(\sigma_1/B_0)^2], \quad (25)$$

$$dB_0/dT = 8\sigma_1 B_0 + 4\sigma_1 \theta_1 + 4(\sigma_1^2/B_0), \quad (26)$$

$$\theta_1 = \frac{1}{2} + 2B_0 - \frac{3}{4}A_0, \quad \sigma_1 = \frac{1}{2}A_0 B_0 - 2B_0^2. \quad (27)$$

Like their parent system (24), Eqs.(25)-(27) have two fixed points, the saddle at  $B_0^- = \frac{1}{2} - \sqrt{c}$ ,  $A_0^- = 4B_0^-$  and a stable focus at  $B_0^+ = \frac{1}{2} + \sqrt{c}$ ,  $A_0^+ = 4B_0^+$ .

According to (5), the soliton's phase  $\chi = \theta + \sigma r^2$  controls the creation and annihilation of the soliton's elementary constituents (whose density is  $|\psi|^2$ ). (If Eq.(2) is used as a model equation for Faraday resonance in granular media or fluids,  $\int |\psi|^2 d\mathbf{x}$  has the meaning of the number of grains or mass of the fluid captured in the oscillon.) Since the creation and annihilation occurs mainly in the core of the soliton, the variable phase component  $\sigma r^2$  plays a marginal role in this process. Instead, the significance of the quantity  $\sigma$  is in that it controls the flux of the constituents between the core and the periphery of the soliton — see the  $\chi_r$ -term in the r.h.s. of (5).

If we perturb the stationary point  $\psi^+$  in the 4-dimensional phase space of (24), the variables  $\theta$  and  $\sigma$  will zap, within a very short time  $\Delta t \sim \frac{1}{\gamma}$ , onto the 2-dimensional subspace defined by the constraints (27). After this short transient the evolution of  $\theta$  and  $\sigma$  will be immediately following that of the soliton's amplitude  $\sqrt{A}$  and width  $1/\sqrt{B}$ . In the case of the  $\psi^+$  soliton, this provides a negative feedback: perturbations in  $A$  and  $B$  produce only such changes in the phase and flux that the new values of  $\theta$  and  $\sigma$  stimulate the recovery of the stationary values of  $A$  and  $B$ . (The phase  $\theta$  works to restore the number of constituents while  $\sigma$  rearranges them within the soliton.) In the case of the  $\psi^-$  the feedback is positive: the perturbation-induced phase and flux (27) strive to amplify the perturbation of the soliton's amplitude and width still further. Finally, for small  $\gamma$  the coupling of  $\theta$  and  $\sigma$  to  $A_0$  and  $B_0$  is via differential rather than algebraic equations. In this case the dynamics of the

phase and flux is inertial and their changes may not catch up with those of the amplitude and width. The feedback loop is destroyed and the soliton destabilises.

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